# Simplicial algebroids and internal categories within $R$-algebroids

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**Abstract**

In this work, by defining Peiffer pairings in the Moore complex of a simplicial algebgroid, we give the close relationship between the category of simplicial algebroids with Moore complex of length 1 and that of internal categories in the category of R-algebroids.

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## 1 Introduction

The notion of crossed module was introduced by Whitehead in [11] as algebraic models for homotopy connected 2-types. The observation that simplicial groups whose Moore complex is of length 1 are equivalent to Whitehead's crossed modules is well known and has led to us to give an algebroid version of this result. Groups are generalised to groupoids, and using this generalisation, Mosa in [9], explored that algebras are appropriately generalised to $R$-algebroids. An $R$-category $A$, [7], is a category equipped with an $R$-module structure on each hom set such that the category composition is an $R$-bilinear map, where $R$ is a commutative ring. It can be seen that an $R$-category is a category which is enriched over a closed category of $R$-modules. G.H. Mosa in his Ph.D. thesis, [9], has studied the notions of $R$-algebroids, crossed modules of $R$-algebroids and some internal categories in the category of algebras. T. Porter, in [10], has defined $R$-algebroids in a bit different way from [9]. Using Porter's definition, an $R$-algebroid $A$ on a fixed set of objects $A_0$, and the hom sets are disjoint family of $R$-modules. He has also proven that in the category of $R$-algebroids over a fixed set, any internal category gives an internal groupoid.

The concept of Peiffer elements in a simplicial group comes from the observation of Brown and Loday's result, [3], about the normalisation of simplical groups. Assume that $\mathbf{G}$ is a simplicial group and $\mathbf{N} = \{N_n\}$ is its Moore complex. Brown and Loday proved that if $G_2$ is generated by degenerate elements, then $\partial_2(N_2) = [\ker d_0, \ker d_1]$. By killing this normal subgroup, it is obtained a crossed module $\partial_2 : N_1 \to N_0$ and an internal category in the category of groups. Arvasi and Porter, in [1, 2], studied Peiffer elements in simplicial commutative algebras, and they applied their results to crossed modules of commutative algebras and categorical algebras and gave a reformulation of Brown Loday's results for commutative algebras.

In this study, using the ideas given by Arvasi and Porter in [1, 2], we give the Peiffer pairings in the Moore complex of a simplicial R-algebroid and explore the close relationship between simplicial algebroids with Moore complex of length 1 and internal categories in the category of R-algebroids by killing some degenarate elements in the Moore complex of a simplicial algebroid. We can say that this result is an algebroid version of the results of Brown-Loday and Arvasi-Porter.

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## 2 Preliminaries

In this section, we give some basic information. We begin, by defining $R$-algebroids and their morphisms. Mitchell, in [7, 8] has given a categorical definition of $R-$algebroids.

**Definition 2.1.** Let $R$ be a commutative ring. An $R-$category $C$ is a category equipped with an $R-$module structure on each hom set such that the composition is $R-$bilinear. An $R$-functor is a functor $\theta : C \to C'$ between $R-$categories such that the maps

$$\ell : C(c_1, c_2) \to C'(\theta c_1, \theta c_2)$$

are $R-$linear. An $R$-category with one object is an associative $R$-algebra.

The following statements can be found in [9].

An $R$-algebroid $C$ is a small $R$-category. Lets elaborate this definition as follows; We shall give the definition of an $R$-algebroid $C$ on a set of objects $C_0$ in the following way.

Recall that $C$ is called a directed graph over a set $C_0$ if there are given functions, $\varepsilon_0, \varepsilon_1 : C \to C_0$, $e : C_0 \to C$ called respectively the source, target and unit maps, such that $\varepsilon_1 e = \varepsilon_0 e = id_{C_0}$. Then, we can write

$$C(x, y) = \{c \in C : \varepsilon_0(c) = x, \varepsilon_1(c) = y\},$$

and if $a \in C(x, y)$, we also write $a : x \to y$.

An $R-$algebroid $C$ is a directed graph over $C_0$ together with for all $x, y, z \in C_0$;

$i)$ an $R-$module structure on each hom set $C(x, y)$,

$ii)$ an $R-$bilinear function, called composition or multiplication,

$$\begin{array}{cccc} \circ : & C(x,y) \times C(y,z) & \longrightarrow & C(x,z) \\ & (a\ ,\ b) & \longmapsto & a \circ b \end{array}$$

The only axioms are that composition is associative, and that the elements $e(x)$, $x \in C_0$ act as identities for composition.

**Example 2.2.** If $C_0$ has exactly one object, then an $R-$algebroid over $C_0$ is an $R-$algebra.

**Example 2.3.** If $C$ is an $R-$algebroid over $C_0$ and $x \in C_0$, then $C(x, x) = C(x)$ is an $R-$algebra.

**Definition 2.4.** If $A$ and $B$ are two $R-$algebroids on the objects set $A_0$ and $B_0$ respectively, the tensor product $A \otimes_R B$ over $A_0 \times B_0$ of the algebroids $A$ and $B$ is the family of $R-$modules

$$\{A(x, y) \otimes_R B(u, v) : x, y \in A_0, \quad u, v \in B_0\}$$

with composition

$$(a' \otimes b') \circ (a \otimes b) = a' \circ a \otimes b' \circ b.$$

### 2.1 $C-$Structures

An extension $\mathbb{E}$ of $C$ by $A$, $C$ and $A$ being both $R-$algebroids on common objects set $C_0$, is a sequence

$$\mathbb{E} : A \xrightarrow{c} B \xrightarrow{\pi} C$$

where $A = \ker \pi$ and each $c \in C$ is the image of some $b \in B$.

If $\mathbb{E}$ is split by $e : C \to B$, $(\pi e = id_C)$, then we say $\mathbb{E}$ is a $C-$structure on $A$.

Let $C$ be an $R-$algebroid on $C_0$. If $x, y \in C_0$, a multiplication $m$ from $x$ to $y$ on $C$ is a pair $(m_L, m_R)$ of natural transformations

$$\begin{array}{llll} m_L : & C(y,-) & \longrightarrow & C(x,-) \\ m_R : & C(-,x) & \longrightarrow & C(-,y) \end{array}$$

such that if $c_1 : t \to x$, $c_2 : y \to z$, one has

$$c_1 \circ m_L(c_2) = m_R(c_2) \circ c_1.$$

Given the multiplications $m : w \to x$, and $n : y \to z$, and $w, x, y, z \in C_0$ then, the multiplications $m, n$ are called permutable if

$$m_L(z)n_R(x) = n_R(w)m_L(y)$$

and

$$n_L(x)m_R(z) = m_R(y)n_L(w).$$

The set of multiplications is said to be permutable if its elements are pairwise permutable. For any $R-$algebroid $C$, let $M(C)$ be a $R-$algebroid with identities. Then, the composition defines a morphism of $R-$algebroids

$$\theta_C : C \to M(C)$$

with $im\theta_C$ a permutable set of multiplications.

### 2.2 Semi-direct product of $R-$algebroids

Given a $C-$structure on $A$, one may get a morphism

$$\theta_K : C \to M(A)$$

with $im\theta_K$ permutable. We shall denote $\theta_R(c)a = a \cdot c$ and $\theta_L(c)a = c \cdot a$. Let $A$ and $C$ be $R-$algebroids and $A$ has a $C-$structure. The semi-direct product of these $R-$algebroids denoted $C \tilde{\times} A$, by

$$(C \tilde{\times} A)(x, y) = C(x, y) \times A(x, y)$$

and

$$(c_1, a_1) \circ (c_2, a_2) = (c_1 \circ c_2, \quad a_1 \cdot c_2 + c_1 \cdot a_2 + a_1 \circ a_2)$$

for $(c_1, a_1) \in C \tilde{\times} A(x, y)$ and $(c_2, a_2) \in (C \tilde{\times} A)(y, z)$.

That this composition is associative follows from the fact that $im\theta$ is permutable. Therefore the semi-direct product $C \tilde{\times} A$ is an $R-$algebroid on objects set $C_0 \times C_0$.

## 3 Simplicial algebroids

A simplicial algebroid $\mathbf{E}$ is a simplicial object in the category of $R-$algebroids, that is, the face and degeneracy maps satisfying the usual simplicial identities and they are identities on objects set and each $E_i$ is an $R-$algebroid on a common objects set denoted by $O$, and the maps $d_i$ and $s_j$

are $R-$algebroid morphisms. A simplicial $R$-algebroid it is useful to give a diagram for a simplicial algebroid as follows:

$$
\begin{tikzcd}
(E,0):\cdots & E_n \arrow[r, dotted, no head] \arrow[d, shift left] \arrow[d, shift right] & E_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] \arrow[d, shift right] & E_1 \arrow[r, shift left, "d_0,d_1"] \arrow[r, shift right] \arrow[l, shift left=4] \arrow[l, shift left=5] \arrow[d, shift left] \arrow[d, shift right] & E_0 \arrow[l, shift left=3, "s_0"] \arrow[d, shift left] \arrow[d, shift right] \\
 & 0 \arrow[r, equal] & 0 \arrow[r, equal] & 0 \arrow[r, equal] & 0
\end{tikzcd}
$$

A Moore complex of a simplicial algebroid $(\mathbf{NE},\partial)$ is a complex of $R-$algebroids, where each $NE_i$ is an $R-$algebroid on a common objects set $O$, and $NE_i$ is defined by

$$NE_i = \bigcap_{i=0}^{n-1} \ker d_i.$$

We shall denote the category of simplicial algebroids by **SimpAlgoid.** We say that the Moore complex $\mathbf{NE}$ of a simplicial algebroid $\mathbf{E}$ is of length $k$ if $NE_n = 0$ for all $n \geq k+1$, so that a Moore complex of length $k$ is also of length $l$ for $l \geq k$. We denote thus the category of simplicial algebroids with Moore complex of length $n$ by $\mathbf{SimpAlgoid}_{\leq n}$.

### 3.1 Peiffer pairings generate for simplicial algebroids

The following terminology and notation is derived from [1, 2].

For the ordered set $[n] = \{0 < 1 < ... < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by;

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various $\alpha_i^n$ by composition. The composition of these generating maps is subject to the following rule: $\alpha_j\alpha_i = \alpha_{i-1}\alpha_j, j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ ... \circ \alpha_{i_r}$ with $0 \leq i_1 < i_2 < ... < i_r \leq n-1$, where the indices $i_k$ are the elements of $[n]$ such that $\{i_1, ..., i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, ..., i_1) : 0 \leq i_1 < i_2 < ... < i_r \leq n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( )denoted by $\varnothing_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, ..., 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

We say that $\alpha = (i_r, ...i_1) < \beta = (j_s, ..., j_1)$ in $S(n)$

if $i_1 = j_1, ..., i_k = j_k$ but $i_{k+1} > j_{k+1}, (k \geq 0)$ or if $i_1 = j_1, ..., i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set.

Now, we shall give the algebroid version of usual Peiffer pairings defined on algebras (or similarly on groups). We shall recall briefly the construction of a family of **R-**linear algebroid morphisms. Let $\mathbf{E}$ be a simplicial algebroid. Define a set $P(n)$ consisting of pairs of elements $(\alpha, \beta)$ from $S(n)$ with $\alpha \cap \beta = \varnothing$ and $\beta < \alpha$ where $\alpha = (i_r, ...i_1), \beta = (j_s, ...j_1) \in S(n)$. The **R-**linear algebroid morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha}(a,b) \otimes NE_{n-\#\beta}(b,c) \to NE_n(a,c) : (\alpha,\beta) \in P(n), n \geq 0\}$$

are given as composites:

$$\begin{array}{rcl} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) & = & p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ & = & p(s_\alpha(x_\alpha) \circ s_\beta(y_\beta)) \\ & = & (1 - s_{n-1}d_{n-1})...(1 - s_0 d_0)(s_\alpha(x_\alpha) \circ s_\beta(y_\beta)), \end{array}$$

where $x_\alpha \in NE_{n-\#\alpha}(a,b)$ and $y_\beta \in NE_{n-\#\beta}(b,c)$, then $s_\alpha(x_\alpha) \in NE_n(a,b)$ and $s_\beta(y_\beta) \in NE_n(b,c)$ and thus, $s_\alpha(x_\alpha) \circ s_\beta(y_\beta) \in NE_n(a,c)$ since degeneracy maps are identity on objects, and

$$s_\alpha = s_{i_r}...s_{i_1} : NE_{n-\#\alpha}(a,b) \to E_n(a,b), s_\beta = s_{j_s}...s_{j_1} : NE_{n-\#\beta}(b,c) \to E_n(b,c),$$

$p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1}...p_0$ with $p_j = 1 - s_j d_j$ for $j = 0, 1, ..., n-1$ and the composition $\circ$ is algebroid composition (or multiplication). Here $O$ is the object set and for any object $a \in O$, we denote $e(a) = 1_a : a \to a$ or only 1.

We will now consider that the algebroid ideal $I_n$ in $E_n$ such that generated by all elements of the form;

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$$

where $x_\alpha \in NE_{n-\#\alpha}(a,b)$ and $y_\beta \in NE_{n-\#\beta}(b,c)$ and for all $(\alpha,\beta) \in P(n)$ and $a,b,c \in O$.

**Proposition 3.1.** Let $\mathbf{E}$ be simplicial algebroid and $n > 0$, and $D_n$ the algebroid ideal in $E_n$ generated by degenerate elements. We suppose $E_n = D_n$, and let $I_n$ be the algebroid ideal generated by elements of the form $C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$ with $(\alpha,\beta) \in P(n)$ where $x_\alpha \in NE_{n-\#\alpha}(a,b), y_\beta \in NE_{n-\#\beta}(b,c)$ with $1 \le r, s \le n$. Then, $\partial_n(NE_n) = \partial_n(I_n)$.

## 4 Simplicial algebroids and internal categories in the category of algebroids

Porter, in [10], has defined an equivalence between the category of internal categories within $R-$algebroids and the category of crossed modules of $R-$algebroids. In fact, this equivalence is the additive form Brown and Spencer [4] theorem about cat-groups and crossed modules. In this section, we will construct an equivalence between simplicial algebroids with Moore complex of length 1, and internal categories within $R-$algebroids. Now, recall that an internal category in the category of $R-$algebroids, denoted by $(D, C, s, t, e, \circ, *)$, is an internal directed graph $(D, C, s, t, e)$ such that $D$ and $C$ are $R-$algebroids on common objects set $O$, and for two operations $\circ, *$ on $D$, the interchange law;

$$(\alpha \circ \beta) * (\gamma \circ \delta) = (\alpha * \gamma) \circ (\beta * \delta)$$

holds whenever either side is defined.

We give the main theorem of this section.

**Theorem 4.1.** The category of simplicial algebroids with Moore complex of length 1 is equivalent to the category of internal categories in the category of $R-$algebroids.

*Proof.* Let $\mathbf{E}$ be a simplicial algebroid. Consider the Moore complex ($\mathbf{NE}$,$\partial$). From this complex, we must have a $C-$structure, and we must construct an internal object. Let $C = NE_0 = E_0$ the

$R-$algebroid, first term in the Moore complex, and $B = E_1$, $A = \ker d_0 = NE_1$. Then, we have a $NE_0-$structure in the following diagram;

$$NE_1 \xrightarrow{i} E_1 \underset{s_0}{\overset{d_0}{\rightleftarrows}} NE_0\ ,$$

where, $d_0 s_0 = id_{NE_0}$. Then, the $R-$algebroid $NE_1$ has a $NE_0-$structure. Therefore, we can define semi-direct product algebroid on the objects set $O \times O$ by

$$(NE_0 \overset{\sim}{\times} NE_1)(x,y) = NE_0(x,y) \times NE_1(x,y)$$

for all $x, y \in O$. Thus, we can define a $NE_0-$structure;

$$NE_1 \xrightarrow{i} NE_0 \overset{\sim}{\times} NE_1 \underset{s}{\overset{\pi}{\rightleftarrows}} NE_0$$

where $i(a) = (0_y^x, a)$, $\pi(c,a) = c$ and $s(c) = (c, 0_y^x)$ for $c \in NE_0(x,y)$ and $a \in NE_1(x,y)$. We will construct from this structure an internal directed graph. We get a diagram;

$$\begin{tikzcd} NE_0 \overset{\sim}{\times} NE_1 \arrow[r, shift left=2, "\varepsilon_0"] \arrow[r, "\varepsilon_1"'] & NE_0 \arrow[l, shift left=3, "e"] \end{tikzcd}$$

where $\varepsilon_0$ is the natural projection given by $\varepsilon_0(c,a) = c$ and $\varepsilon_1(c,a) = c + \partial_1 a$, and $e$ is the section by $e(c) = (c, 0_y^x)$. This internal directed graph can be given by a partially defined composition as follows

$$(c,a) * (c',a') = (c, a + a'),$$

where, $\varepsilon_0(c',a') = c' = c + \partial_1 a = \varepsilon_1(c,a)$, and $(c,a),(c',a') \in (NE_0 \overset{\sim}{\times} NE_1)(x,y)$ and $\partial_1 : NE_1 \to NE_0$ is a map restricted to face map $d_1$. Not that $e(c)$ acts as a left identity with $e(c + \partial_1 a)$ as right identity, i.e., we almost have a category object in the category of $R-$algebroids. The second operation " $\circ$ " on

$$NE_0 \overset{\sim}{\times} NE_1$$

is given by

$$(c_1, a_1) \circ (c_2, a_2) = (c_1 \circ c_2, a_1 \cdot c_2 + c_1 \cdot a_2 + a_1 \circ a_2)$$

for $(c_1, a_1) \in \left(NE_0 \overset{\sim}{\times} NE_1\right)(x,y)$ and $(c_2, a_2) \in \left(NE_0 \overset{\sim}{\times} NE_1\right)(y,z)$, where $a_1 \cdot c_2 = a_1 \circ s_0(c_2)$ and $c_1 \cdot a_2 = s_0(c_1) \circ a_2$.

Associativity is immediate; in fact the only thing to check is that $*$ is a morphism of $R-$algebroids, i.e., that the interchange law

$$(\alpha \circ \beta) * (\gamma \circ \delta) = (\alpha * \gamma) \circ (\beta * \delta)$$

holds whenever either side is defined.

If $(c,a) \in (NE_0 \overset{\sim}{\times} NE_1)(x,y)$, then $c+\partial_1 a$ also is from $x$ to $y$, since $d_1$ is the identity on objects, so $(c,a),(c+\partial_1 a,a') : x \to y$ in $NE_0 \overset{\sim}{\times} NE_1$ and $(d,b),(d+\partial_1 b,b') : y \to z$. Since, the length of Moore complex is 1, we have following

$$\partial_2(NE_2) = 0.$$

Therefore, for $a \in NE_1(x,y)$ and $b \in NE_1(y,z)$, we can write $a \circ b - a \circ s_0 d_1(b) \in \partial_2(NE_2)$ and since $\partial_2(NE_2) = 0$ we have

$$a \circ b - a \circ s_0 d_1(b) = 0_z^x,$$

or

$$a \circ b = a \circ s_0 d_1(b).$$

By using these statements, we calculate;

$$\begin{aligned}
(\alpha \circ \beta) * (\gamma \circ \delta) &= [(c,a)\circ(d,b)] * [(c+\partial_1 a,a')\circ(d+\partial_1 b,b')] \\
&= [(c \circ d, \quad a \circ s_0(d) + s_0(c)\circ b + a \circ b)] \\
&\quad *[((c+\partial_1 a)\circ(d+\partial_1 b)\,, \quad a' \circ s_0(d) + a' \circ s_0 d_1(b) + \\
&\quad s_0(c)\circ b' + s_0 d_1(a) \circ b' + a' \circ b')] \\
&= (c \circ d, \quad a \circ s_0(d) + s_0(c) \circ b + a \circ b + a' \circ s_0(d) + \\
&\quad a' \circ s_0 d_1(b) + s_0(c) \circ b' + s_0 d_1(a) \circ b' + a' \circ b') \\
&= (c \circ d, \quad a \circ s_0(d) + s_0(c) \circ b + a \circ b + a' \circ s_0(d) + \\
&\quad a' \circ b + s_0(c) \circ b' + a \circ b' + a' \circ b') \quad \text{since } \partial_2(NE_2) = 0 \\
&= (c \circ d, \quad (a+a') \circ s_0(d) + s_0(c) \circ (b+b') + (a+a') \circ (b+b')) \\
&= (c, a+a') \circ (d, b+b') \\
&= [(c,a) * (c+\partial_1 a, a')] \circ [(d,b) * (d+\partial_1 b, b')] \\
&= (\alpha * \gamma) \circ (\beta * \delta).
\end{aligned}$$

Thus, we have an internal category in the category of $R-$algebroids,

$$(NE_0 \overset{\sim}{\times} NE_1, NE_0, s, t, e, \circ, *).$$

Now, we can define a functor from the category of simplicial algebroids with Moore complex of length 1, to the category of internal categories in the category of $R-$algebroids;

$$\Theta : \mathbf{SimpAlgoid} \to \mathbf{IC}_R.$$

Here $\mathbf{IC}_R$ is the category of internal categories in the category of $R-$algebroids.

Conversely, let $(D, C, s, t, e, \circ, *)$ be any internal category within $R-$algebroids. Let $A = \ker s$, then, we have

$$A \longrightarrow D \underset{e}{\overset{s}{\rightleftarrows}} C\ .$$

This gives a $C-$structure on $A$, explicitly we have

$$c \cdot a = e(c) \circ a$$

and

$$a \cdot c = a \circ e(c).$$

Let $E_0 = C$ and $E_1(x,y) = C \tilde{\times} A(x,y)$, where $D$ is isomorphic to $C \tilde{\times} A$ by the isomorphism;

$$\begin{array}{rcl} \Theta : D & \to & C \tilde{\times} A \\ d & \longmapsto & (s(d), d - e(s(d))) \end{array}$$

We define face and degeneracy maps in the following way, between $E_1(x,y) = (C \tilde{\times} A)(x,y)$ and $E_0(x,y) = C(x,y)$;

$$d_0(c,a) = c + t(a)$$

$$d_1(c,a) = c$$

and

$$s_0(c) = (c, 0_y^x)$$

for all $c \in C(x,y)$ and $a \in A(x,y)$. It can be easily showed that these maps satisfy the usual simplicial identities. Thus, we have a $1-$truncated simplicial algebroid with the morphisms

$$\begin{tikzcd} C \tilde{\times} A \arrow[r, shift left=2, "d_0{,}d_1"] \arrow[r] & A \arrow[l, shift left=2, "s_0"] \end{tikzcd}$$

There is a coskeleton functor $\mathbf{Cos}_k$, from the category of $k-$truncated simplicial algebroids to simplicial algebroids. Consequently, we get the following diagram

$$\begin{tikzcd} \mathbf{SimpAlgoid} \arrow[rr, "\Theta"] & & \mathbf{IC}_R \arrow[dl] \\ & \mathbf{Tr}_1\ \mathbf{SimpAlgoid} \arrow[ul, "\mathbf{Cos}_k"] & \end{tikzcd}$$

and this enables us to define a functor

$$\mathbf{S} : \mathbf{IC_R} \to \mathbf{SimpAlgoid}.$$

It can be easily verified that the length of the Moore complex of this simplicial algebroids is 1, and we see that the compositions $\mathbf{S\Theta}$ and $\mathbf{\Theta S}$ are identity. Thus, the proof is complete. Q.E.D.

Consequently, we showed that the category of internal categories in the category of $R$-algebroids is equivalent to that of simplicial algebroids with Moore complex of length 1.

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